A Rebuttal to the Statement: “Missing, Delayed, or Muddled Topics in Common Core’s Math Standards” by R. James Milgram and Ze’ev Wurman

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Standards are negotiated agreements among experts and are designed to reflect diverse knowledge and experience. The CCSS-M Standards writing involved a variety of experts over 18 months of hard work. With this in mind, I seek to respond directly to the claims in the document provided to the Commission by R. James Milgram and Ze’ev Wurman. As a Member of the National Validation Committee and a citizen of Raleigh, North Carolina, I submit this statement in order to correct the record and challenge, with evidence, a number of inaccurate or misleading statements in Milgram/Wurman document. My response is limited to grades K-8 to illustrate to the average reader the kinds of spurious reasoning in the document. There are six sections to the critique.

I. On Using Standard Algorithms

Developing fluency in the standard algorithms is addressed explicitly and repeatedly in the standards. Using a sound developmental approach, a conceptual base is built for an operation using models and place value; then in the following years, fluency in the use of standard algorithms (with cases with more and more digits) is required explicitly of our students.

- Fluency in addition and subtraction is addressed in standards 2.NBT.B.5, 3.NBT.A.2, and 4.NBT.B.4
- Fluency in multiplication is addressed in 3.OA.C.7, and 5.NBT.B.5
- Fluency in division is addressed in 6.NS.B.2
- Fluency in all four operations for rational numbers is addressed in 6.NS.B.3

Therefore, Milgram/Wurman’s claim, “automaticity with the standard algorithms of arithmetic not required in CCMS,” is simply untrue. Singapore uses the language “committing to memory;” hence, Milgram/Wurman would discredit the Singapore standards for lacking the words “proficiency,” “mastery,” or “automaticity.” Likewise Finland, another high performing country, would also be discredited. In fact, our students are required to learn the standard algorithms, quite properly, to fluency.

The relevant standards for each of the four operations are reproduced below.

2.NBT.B.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

3.NBT.A.2 Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

3.OA.C.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

4.NBT.B.4 Fluently add and subtract multi-digit whole numbers using the standard algorithm.

1 All statements are the professional opinion of the author and do not represent any official organization or institution.
5.NBT.B.5 Fluently multiply multi-digit whole numbers using the standard algorithm.
6.NS.B.2 Fluently divide multi-digit numbers using the standard algorithm.
6.NS.B.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

II. On Ratio and Proportional Relationships in Grades 6 and 7

Milgram/Wurman claim that ratio is introduced too late, but they overlooked standards placed carefully in grades 4 and 5 that build a foundation for an understanding of ratio. To understand ratio and proportion in CCSS-M, one has to understand a basic learning concept—the learning trajectory/progression, which describes, based on empirical study, how students come to understand an idea, moving from naïve or partial ideas to more advanced ones. Milgram/Wurman are correct that the word “ratio” is not introduced in K-5. However, two standards, one in fourth (4.MD.A.1) and one in fifth grade (5.MD.A.1), explicitly address measurement conversion within measuring systems (inches to feet), which constitutes a substantial introduction to ratio reasoning. Moreover, another fifth-grade standard, 5.OA.B.3, has students examine two-column tables of (x,y) and graph them to explore their relationships. One such table of would likely be based on a multiplicative relationship (1,3), (2,6) (3,9) etc. all examples of the ratio 3:1. Thus, these three grade 4 and 5 standards, in measurement and in operations and algebraic thinking, ensure that students are ready for the extensive treatment of ratio in sixth grade. The relevant standards are reproduced below:

4.MD.A.1 Solve problems involving measurement and conversion of measurements. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb., oz.; 1, ml; hr., min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...

5.MD.A.1 Convert like measurement units within a given measurement system. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

5.OA.B.3 Analyze patterns and relationships. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

The authors also express indignation on the CCSS-M’s use of the word “nearly” in the example introducing ratios that states: "For every vote candidate A received, candidate C received nearly three votes." Perhaps their mathematical preference is for an exact ratio. However, students live in a world in which they need to apply ratio to the world around them as a model for relationships. If one candidate received 35,415 votes and the other received 102,745, it is certainly appropriate to describe it as written above, as one being “nearly” three times as many. The “nearly” in the sentence refers to the approximation or estimation made. Approximation and estimation can and should be taught at early grades; hence it is therefore inaccurate for Milgram/Wurman to state, “At best it [likely] corresponds to a range of ratios, but the tools for handling such objects are not covered until college and require advanced calculus.” Skills in approximation and estimation are part of the fundamental mathematical practices of modeling with mathematics (Practice 4) and attending to precision (Practice 6) addressed across the grades.

With further regard to ratio and proportion, Milgram/Wurman’s claim that high achieving countries begin with ratios using the following definition is not true. They wrote, in high-achieving countries, students are first given the definition: two points in the coordinate plane, (a, b) and (c, d), are in a proportional relationship if and only if neither is (0,0) and they both lie on a single straight line through the origin. Presuming that a is non-zero, then writing b = ra (so r =
In fact, in most countries, no definitions are included in the standards, just statements of topics and when to teach them. Milgram/Wurman’s definition is in the form of a theorem, is not at an introductory level at all, and is an issue of curriculum not standards. This confusion by the writers demonstrates their lack of experience in teaching children. The reason standards are written by coalitions of experts—including mathematicians, math educators, special educators, professionals with other areas of expertise—is to help maintain the important distinctions among standards, curricular approaches, and formal definitions.

The difference between how the CCSS-M introduces graphing and ratios in 6.RP.A.3.A and 7.RP.A.2.A and Milgram/Wurman’s suggested approach is really quite understandable. In CCSS-M, one builds a ratio relationship by drawing a line through the origin and a point \((a, b)\) to establish the ratio \(b:a\). In Milgram/Wurman’s definition of proportionality represents a test for a proportional relationship, given two points. One checks to see if a line through those two points only goes through the origin. Either approach is mathematically correct. Developmentally, however, it makes sense to learn a concept of ratio first, finding out that a set of equivalent ratios written as an ordered pairs using a ratio table falls along a straight line through the origin.

Learning to test for proportional relations is important, but it comes later. The relevant CCSS-M standards are written below:

6.RP.A.3.A Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

7.RP.A.2.A Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

Three crucial points help one to avoid being confused or misled by the statements in the Milgram/Wurman document:

1. Standards are not intended as statements about how to teach a topic; they are instead a means to coordinate what to teach at a given grade,

2. Standards are not formal mathematical definitions and theorems (though they depend on them).

3. Standards should be ordered by grade to form coherent paths of learning, or learning trajectories/progressions (which are found by careful empirical study), not simply made up as logical thought experiments by mathematicians or educators.

III: On Other Incorrect or Misleading Statements

1. Milgram/Wurman incorrectly claim that “CC fails to teach decimals until grade 4, about two years behind high-achieving countries.” Finland groups its standards into grade bands of 3-5 so that one cannot tell when decimals are introduced. Singapore introduces decimals in grade 4. No country I could find introduces decimals in second grade.

2. Milgram/Wurman falsely claim that “CC excludes conversion between [among] different forms of fractions: regular fractions, decimals, and percents.” While Milgram/Wurman wanted the skill written in one place, the authors of the Standards chose to discuss this topic gradually over three relevant standards. In sixth grade (6.RP.A.3.C), the Standards address percent in relation to ratios and rates thus handling the conversion from \(\frac{a}{b}\) to \%; in seventh grade (7.NS.A.2.D), with converting a fraction to a decimal covered in fourth and fifth grades, the Standards concentrate on converting a fraction to a decimal by division; In eighth grade (8.NS.A.1), the CCSS-M standards discuss how to convert a repeating decimal to a fraction. This
It seems clear that recognizing the learning progression proved elusive to Milgram and Wurman. But it leads them to misrepresent the claim and misuse the word “exclude.” The relevant standards are reproduced below:

6.RP.A.3.C Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

7.NS.A.2.D Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

8.NS.A.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion, which repeats eventually into a rational number.

3. The failure of these authors to sufficiently understand known patterns in student learning is further illustrated by their comment about a “minor mathematical error” in these standards. The “minor” mathematical error to which they refer is that decimals that terminate in zero can be classified as repeating decimals, because, for example, .5 could be written as .5000…, a decimal with a repeating zero.

Formally, this is a true statement. However, to a student, a decimal that terminates is fundamentally different than one that repeats infinitely. The idea that 1/3 is located one third of the way from 0 to 1 yet is exactly equal to .333… typically amazes students. Students are often delighted to learn that a number that appears to grow infinitely can be located precisely at a point on the number line and named, for example, 1/3. Seeing that 1/3 equals .5000 does not produce the same compelling discussions. Distinguishing repeating decimals and decimals terminating in 0 was not an error: the authors of the Standards were quite right to separate these two ideas.

4. Milgram/Wurman’s discussion of compound interest shows a similar failure of experience with student learning. I taught for fourteen years at Cornell University; I studied students’ (including college freshmen) understanding of exponential functions, and I can assure you that prior to CCSS-M, a multitude of smart students failed to understand compound interest due to memorizing instead of understanding formulas like those presented. When told a person earns 5% compound interest on $100 over three years, for instance, these Ivy league students would inevitably multiply .05 x .05 x .05 and then multiply by $100. Their failure can be traced back to a failure to understand percent increase as multiplication by single value, as, in this case, $100 x 1.05 rather than to solve it as .05 x 100 + 100. The two-step process fails them as they tackle compound interest. Understanding the importance of first establishing how to understand percent increase or decrease by introducing exponents using Scientific Notation as in CCSS-M provides a better chance of avoiding this misconception. CCSS-M addresses percent increase and decrease three ways, in standards 7.RP.A.3, 7.EE.A.2, and 7.EE.B.3 (reproduced below).

7.RP.A.3 Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.

7.EE.A.2 Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. For example, a + 0.05a = 1.05a means that “increase by 5%” is the same as “multiply by 1.05.”

7. EE.B.3 Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. For example: If a woman making $25 an hour gets a 10% raise, she will make an
additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.

IV: On the Discussion of Rigor

Milgram and Wurman cited “the Dictionary of Education Reform (edglossary.org/rigor) as defining “rigor” as, “While dictionaries define the term as rigid, inflexible, or unyielding, educators frequently apply (the terms) rigor or rigorous to assignments that encourage students to think critically, creatively, and more flexibly. Likewise, they may use the term rigorous to describe learning environments that are not intended to be harsh, rigid, or overly prescriptive, but that are stimulating, engaging, and supportive.” Milgram and Wurman used this citation but failed to provide the actual definition that the authors said applies to using rigor to describe standards from later the same source. It states, “One common way in which educators do use rigor to mean unyielding or rigid is when they are referring to “rigorous” learning standards and high expectations—i.e., when they are calling for all students to be held to the same challenging academic standards and expectations.” (same citation)

Based on their misrepresentation of how educators use the term “rigor,” they conclude At best, education professionals who use “rigorous” to describe CCMS may well be saying that its standards promote “creativity and flexible thinking” (although they do not indicate how) and they may also be implying absolutely nothing about accuracy or intellectual demand.” This discussion of rigor is frankly polemical, and demeans the hundreds of us who dedicated a year and a half of our lives voluntarily to developing the CCSS-M.

Rather, the CCSS-M were designed to provide coherence, focus, and rigor. Coherence refers to how they fit together, including the underlying trajectories/progressions. Focus refers to the difficult decisions to eliminate many topics at each grade level, to keep them concise. And rigor refers to the careful attention to matching our standards to other relevant sets of international standards.

Furthermore, in the next paragraph, they say, “…for foundational mathematics, neither creativity nor flexibility is desired.” This comment again illustrates their complete inability to understand or appreciate the beauty in children’s thinking about foundational math, which is often creative and quite clever. It seems to suggest that flexibility and creativity in mathematics can only be demonstrated by advanced mathematicians.

V: All Standards Require a Means of Review and Improvement

Any standards writing process results in some less-than-optimal statements. For instance, I agree with Milgram and Wurman that prime factorization should have been explicitly mentioned in the CCSS-M. However, it should be noted that Standard 4.0A.B.4 does address primes and composites. GCF and LCM are addressed in sixth grade; these topics are most efficiently addressed using prime factorization. It would be simple to modify the CCSS-M to include explicit reference to prime factorization. Everyone connected with the Standards expected to see minor modifications—as well as to see changes over time as tools, curriculum, disciplines and applications change. Many of us would welcome an opportunity to reexamine and revise a few of the CCSS-M standards through a systematic process to make periodic revisions.

The assessments are too often confused with the Standards. It is critical to recognize that whatever assessment system is selected will provide an important source of data to guide the revision of the Standards. The process of Standards review and revision must be connected to the assessment system so as to create a means to gauge progress over time, make appropriate
adjustments by panels of experts, and, ultimately, to improve coherence and alignment of Standards, instruction, and assessment.

VI: Conclusions

The NC Legislature decided to review their decision to adopt CCSS after three years of statewide preparation for implementation. This decision has left our teachers in NC with massive ambiguity, when their focus should be on successful and thorough implementation that leverages the economies of scale created by the large coalition of adopting States. This is unfortunate because there is much work to do in this regard, and our international competitors are getting the job done.

The Validation Committee recently reviewed the critiques of the Standards; 23 of 24 members who initially validated the Standards signed a letter dated March 5, 2015 and sent to your Commission on Friday (available at the Collaborative for Student Success, through chris@forstudentsuccess.org) to endorse the process of Standards creation, and the Standards’ quality in relation to those of other countries. I earnestly urge this Commission to endorse the CCSS Standards and let our students share in the benefits of improving the learning of mathematics across the great State of North Carolina.